STABILITY OF DEFORMATIONS OF ELASTIC RODS

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Summary

Mechanics is a fascinating field of physics. Its main purpose is to study the interaction of particles in the presence of forces such as gravity. In contrast to the dynamics of particles, the study of continuous media, such as elastic bodies, is challenging and still today not entirely well-understood. This is due to the fact that continuous media behave in numerous and complex ways. For example it is difficult to describe mathematically the twisting and bending of elastic rods but approximate models do exist.

The study of the dynamics of elastic rods is a field of research that is interesting in itself because it gives rise to beautiful and complex mathematical structures. Furthermore, understanding elastic rods is crucial for applications in several domains of science.

In this proposal I will focus on a model that has been proven to be quite successful in describing the dynamics of rods in a certain approximation: the set of Kirchhoff equations. These equations are differential equations, that is they are equations involving derivatives. The solutions to these equations represent possible behaviors for elastic rods.

The goal of the proposal is to study two types of solutions to the Kirchhoff equations: the periodic and soliton solutions. These two types of solutions have been proven to be very useful in other fields of science such as optics and fluid mechanics. More precisely, the goal of the proposal is to study the stability properties of solutions of the Kirchhoff equations. Stability is a fundamental concept in physics. An illustration of this concept is given by trying to make a pencil stand on its lead. In theory, it is possible but, in practice, because it is such an unstable state, it cannot be done. The same concept applies to solutions of differential equations: some are stable and some are not. In the case of differential equations though, it is often necessary to develop some sophisticated mathematical methods to study stability. However, it is crucial to study the stability of the solutions of the Kirchhoff equations because only stable solutions can be observed experimentally.

The research of this proposal can be summarized as followed:

- Develop a method to study the stability of certain solutions in the context of elasticity.
- Apply this method to the equations I am studying to obtain the stable solutions.
- Study the physical conditions under which these stable solutions propagate in a rod.

1. Introduction

Classical mechanics is a domain of physics whose goal is to describe mathematically the dynamics of particles in the presence of forces such as gravity. For example, laws of mechanics can be used to obtain the trajectory of a baseball thrown upward or a planet orbiting around the sun. The basic equations for classical mechanics were discovered by Newton in the seventeenth century. Since then, the community of physicists and mathematicians have developed techniques to describe more complex phenomena. One great challenge in this field has been, and still is today, the description of the dynamics of continuous media, such as fluids and elastic bodies. This is due to the fact that continuous media behave in numerous and complex ways. Even today, they can only be described approximatively.

The goal of the proposed research deals with the study of elastic bodies and, more precisely, elastic rods. The most common examples of elastic rods as they are understood in this proposal are cables and hoses. The study of the dynamics of elastic rods is a field of research that is interesting in itself because it gives rise to beautiful and complex mathematical structures. Furthermore, understanding elastic rods is important for applications in different domains of science. In biology, rod models are used to describe the coiling of different structures such as the DNA [?, ?]. In engineering, they are used to study the behavior of submarine cables [?]. In physics, the study of twisted rods is useful in several applications: in hydrodynamics for describing of the motion of vortex tubes [?], in astrophysics for describing the formation of sun spots and the heating of the solar corona [?], and in the study of polymers [?].

The commonly used equations for describing elastic rods dynamics were found by **Kirchhoff** in the beginning of the twentieth century [?]. These equations are differential equations, that is they are equations involving derivatives. The **solutions** to these equations represent behaviors for an elastic rod. For example, figure 1 represents the behavior of an elastic rod found by studying solutions to the Kirchhoff equations. One goal of the proposed research is to study important classes of solutions of these equations.

Not all the solutions of the Kirchhoff equations can be realized experimentally. As I will explain in Section 2, only **stable** solutions can be observed. My second goal is to develop methods to study the stability properties of the solutions.

The remaining of the proposal is organized as follows. In Section 2, the notion of stability and its importance in the context of elastic rods are explained. In Section 3, solutions to the Kirchhoff equations are described. Finally, in Section 4, the goals of the proposed research are thoroughly explained.

2. Stability

Stability is a fundamental concept in physics. An illustration of this concept is given by trying to make a pencil stand on its lead. In theory, it is possible but, in practice, because it is such an unstable state, it cannot be done.

In the example just described, the study of stability is very simple and there is no need for a mathematical analysis to prove or disprove the stability of the system. The concept of stability carries over to solutions of differential equations such as the Kirchhoff equations. In this context, the concept of stability takes an abstract form and its study often involves some sophisticated mathematical tools. However, such a study is fundamental because only stable solutions can be realized experimentally. Unstable solutions, just like in the case of the pencil explained above, although exist in theory will never be observed. Hence, with

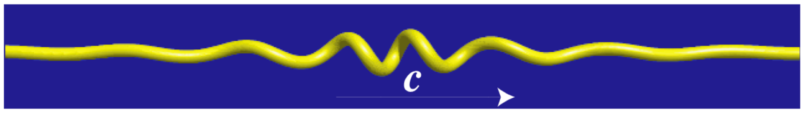

FIGURE 1

stability studies, one can distinguish the solutions that can be realized experimentally from the ones that cannot.

3. Solutions

There are two types of solutions that will be considered: soliton and periodic ones.

Soliton solutions (also called solitary wave solutions) were first observed as water waves. In this context, they take the form of a single wave that propagates at a certain speed. What distinguishes them from other common waves is that they occur as one solitary wave that is very persistent and, unlike other waves that dissipate rapidly, can be seen traveling for miles and miles. The first sighting of a soliton was made by J.S. Russell in 1834 who followed for miles the strangely persistent solitary wave that had formed in a canal in Scotland. This observation gave rise, several years later, to the mathematical field of integrability in which the concept of soliton plays a major role. Another example of a soliton is given by the recent tsunamis that hit Asia few weeks ago. A soliton solution in the context of an elastic rod is illustrated in Figure 1 in which a localized perturbation of the rod propagates to the right.

Periodic solutions are more commonly experienced in nature. For example, as water waves, they take the form of ripples usually seen at the surface of water when the wind is blowing. Periodic solutions have applications in several domain of science but particularly in optics where signals (in optical fibers for instance) can often be described by periodic structures.

4. Methods and goals of the project

The main goal of the project is to develop mathematical methods to study solutions and their stability properties in the context of the dynamics of elastic rods. In some of my previous previous articles [**?**, **?**, **?**] in which I have studied soliton solutions, two types of results were obtained. First, mathematical methods were developed to study the stability of soliton solutions propagating in elastic rods. These new methods will be extended to other fields of science. Second, it was possible to find the stable solutions and thus know what kind of soliton solutions can be observed experimentally.

This research proposal deals with the other important case: periodic solutions. Two types of results will be obtained. First, a method will be developed to study the stability of periodic solutions found in models of elastic rods. Although periodic solutions were shown to be crucial in many applications in physics and mathematics, a very small amount of work in the literature is devoted to their stability properties. Second, using these methods, periodic solutions that are stable will be obtained and studied.

It is important to note that the equations I am considering are not the Kirchhoff equations *per se* but rather a simplification derived by Goriely and coworkers (see for example [**?**]) who studied the amplitude equations of the dynamics of a twisted rod beyond the threshold of its first writhing instability. The equations in question take the form

$$\frac{\partial^2 A}{\partial t^2} - c_0^2 \frac{\partial^2 A}{\partial x^2} = \mu A - A|A|^2 + A\frac{\partial B}{\partial x},$$

(1)

$$\frac{\partial^2 B}{\partial t^2} - \frac{\partial^2 B}{\partial x^2} = -\frac{\partial |A|^2}{\partial x}.$$

This system does admit both periodic and soliton solutions.

The method I will be developing is based on the mathematical tool called **Evans function** [**?**]. This method was initially introduced to study the stability of soliton solutions in contexts other than the theory of elasticity. To understand the basic idea of Evans functions, it is necessary to perform a linearization of (**??**) about a solution. This gives rise to the equality

$$\mathcal{L}\, w = \lambda\, w,$$

where $\mathcal{L}$ is a matrix linear differential operator, λ and w are, respectively, the eigenvalue and eigenvector associated to the problem. It is the eigenvalue that tells us if a solution is stable or not. In the case where the solution is a soliton, the Evans function can be used to find information about the eigenvalue and thus about stability (there is a great amount of literature on this subject; see [**?**] for a recent review). In the case where the solution is periodic, investigating stability is more challenging and much less work have been done on the subject (see [**?**] for example). The main result that will enable me to develop my method can be found in [?] and takes the form of a theorem that says that the eigenvalues for the periodic solutions can be related in some way to the eigenvalues of the solitons solutions. In my previous work [?, ?, ?], I have been studying the eigenvalues of the soliton solutions by means of the Evans function technique. I thus believe it will be possible to combine my previous results with the theorem mentioned above and develop a method to study the stability of the periodic solutions.

Once the stability properties of the solutions will be known, a study of the physical implications will be performed. On the one-hand, stable solutions will be related to behaviors of rods and, on the other hand, the physical conditions under which these stable periodic solutions propagate in a rod will be studied.

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