STABILITY PROPERTIES OF LIGHT PROPAGATING IN FIBER OPTICS

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Summary

The study of the propagation of light is a fascinating fields of research that is interesting in itself because it gives rise to beautiful and complex mathematical structures. Furthermore, understanding the propagation of light is crucial in applications such as lasers and optical fibers.

In this proposal I will focus on a model that has been proven to be quite successful in describing the propagation of light in a certain type of fiber optics: the Manakov system. This system consists of two *differential equations*, that is two equations involving derivatives. The solutions to these equations represent possible behaviors for the light.

The goal of the project is to study solutions to the Manakov system which are called solitons. This type of solutions has been proven to be very useful in several fields of science. In the context of fiber optics, the hope is that solitons can provide a very efficient way to send information on long distances through fiber optics. More precisely, the project deals with the study of the stability properties of soliton solutions. Stability is a fundamental concept in physics. An illustration of this concept is given by trying to make a pencil stand on its lead. In theory, it is possible but, in practice, because it is such an unstable state, it cannot be done. The same concept applies to solutions of differential equations: some are stable and some are not. In the case of differential equations though, it is often necessary to develop some sophisticated mathematical methods to study stability. However, it is crucial to study the stability of the solutions of the Manakov system because only stable solutions can be observed experimentally.

Additionally, I intend to study generalizations of the Manakov system which take into account additional physical properties of the fiber optics. I plan to use the results obtained for the Manakov system and use them to study (by a perturbative approach) these more general systems.

The research of this proposal can be summarized as followed:

- Develop a method to study the stability of certain solutions in the context of optics.
- Study the physical conditions under which stable solutions of the Manakov system propagate.
- Extend the study to systems that include more physical effects than the Manakov system.

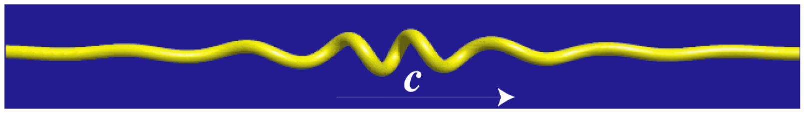

FIGURE 1

1. Introduction

Optics is a domain of physics which studies, among other things, the propagation of light. The basic equations for describing the electro-magnetic waves that constitute light were discovered by James Clerk Maxwell in the nineteenth century. Since then, the community of physicists and mathematicians have developed techniques to use Maxwell's equations to describe more and more complex phenomena. One of the most active area of study in this field today the description of the propagation of light in optical fibers.

The goal of the project is to study one of the most important system of equations used to describe the propagation of light in optical fibers. It is called the Manakov system [29]. This system consists of two equations which are differential equations, that is they are equations involving derivatives. The **solutions** to these equations represent behaviors for the propagating light.

Not all the solutions of the Manakov system can be realized experimentally. As I will explain in Section 2, only **stable** solutions can be observed. The main goal of the project is to develop methods to study the stability properties of the solutions of the Manakov system itself and generalizations of it.

The remaining of the proposal is organized as follows. In Section 2, the notion of stability and its importance in the context of light propagation are explained. In Section 3, the concept of integrability, which plays a central role in the project, is explained. In Section 4, soliton solutions to the Manakov equations are described. Finally, in Section 5, the goals of the proposed research are thoroughly explained.

2. Stability

Stability is a fundamental concept in physics. An illustration of this concept is given by trying to make a pencil stand on its lead. In theory, it is possible but, in practice, because it is such an unstable state, it cannot be done.

In the example just described, the study of stability is very simple and there is no need for a mathematical analysis to prove or disprove the stability of the system. The concept of stability carries over to solutions of differential equations such as the ones forming the Manakov system. In this context, the concept of stability takes an abstract form and its study often involves some sophisticated mathematical tools. However, such a study is fundamental because only stable solutions can be realized experimentally. Unstable solutions, just like in the case of the pencil explained above, although exist in theory will never be observed. Hence, with stability studies, one can distinguish the solutions that can be realized experimentally from the ones that cannot.

4. Integrability

Integrability is a property that characterizes systems for which one can make long term predictions. For example, the equations describing the motion of a satellite around a planet are integrable because one can use them to predict with accuracy where will the satellite be at any point in the future. Integrability can be seen as the complete opposite of what is called chaos. Chaos characterizes the systems which are extra sensitive to initial conditions, and thus long term predictions is practically impossible. The *butterfly effect* is a popular notion which illustrates what chaos means in the context of the weather system. It can be stated as follows: a butterfly's wings in Brazil might create tiny changes in the atmosphere that ultimately cause a tornado to appear in Texas.

The great advantage of studying the integrable systems, is that, due to their high degree of regularity, it is usually possible to perform a mathematical analysis to understand fully their behavior. However, integrable systems are usually idealized and it is often necessary to alter an integrable equation in order to be able to take into account all the effects encountered in a given experiment. In the example of a satellite orbiting around a planet for example, the effect that would break integrability can take the form of the friction between the satellite and the atmosphere. The study of the idealized systems is nevertheless still very important in the context of applications. The main reason is that the study of the idealized integrable models gives a lot of insight on the behavior of the more realistic and complex models.

The models this project is dealing with take the form of differential equations. In this context, the immediate consequence of integrability is the presence *soliton solutions* which we describe below.

3. Soliton Solutions

Soliton solutions (also called solitary wave solutions) were first observed as water waves. In this context, they take the form of a single wave that propagates at a certain speed. What distinguishes them from other common waves is that they occur as one solitary wave that is very persistent and, unlike other waves that dissipate rapidly, can be seen traveling on large distances. The first sighting of a soliton was made by J.S. Russell in 1834 who followed for miles the strangely persistent solitary wave that had formed in a canal in Scotland. This observation gave rise, several years later, to the mathematical field of integrability in which the concept of soliton plays a major role. Another example of a soliton is given by the tsunamis that hit Asia few years ago. Soliton are now known to appear in a wide variety of contexts. A soliton solution in the context of an elastic rod is illustrated in Figure 1 in which a localized perturbation of the rod propagates to the right.

One of the most interesting applications of soliton solutions lies in the field of optical science. Indeed, like water, light travels in the form of waves (more precisely, electro-magnetic waves). Thus, in a very similar way as with water, it is possible for the light to travel in the form of a soliton. Engineers are very interested in using solitons in optical fibers because it would provide a very efficient way to send information on large distances.

4. Methods and goals of the project

The main goal of the project is to develop mathematical methods to study solitons and their stability properties in the context of fiber optics. More precisely, I will be studying solutions to what is called the Manakov system which is well-known to describe the propagation of light in certain types of optical fibers [30]. The Manakov system is a nonlinear system of partial differential equations that take the following form

$$(1)\quad \begin{aligned} &iA_t + A_{xx} + (|A|^2 + |B|^2)A = 0 \\ &iB_t + B_{xx} + (|B|^2 + |A|^2)B = 0 \end{aligned}$$

The variables A and B depend on the position variable x and time variable t.

In order to study the stability of a solution of the Manakov system (1), it is necessary to perform a linearization of (1) about a solution. This gives rise to the equality

$$(2)\quad \mathcal{L}\, w = \lambda\, w,$$

where $\mathcal{L}$ is a matrix linear differential operator, λ and w are, respectively, the eigenvalue and eigenvector associated to the problem. The set of values the eigenvalue λ can take tells us if a solution is stable or not.

The Manakov system (1) is integrable and thus, as mentioned before, it is possible to use mathematics to understand it quite well. In particular, to perform the study of stability properties of its solutions, people in the past have used existing tools such as what is called *Hamiltonian formalism* (see for example [15, 31, 32]). This method takes advantage of the fact that the system conserves energy in order to study the values of the eigenvalue λ in (2). However, there are two drawbacks to this approach. First, in order to use Hamiltonian formalism to study the stability, the solutions must have a certain specified form which is not always the case for the Manakov system (see [10], in which the form the solutions are restricted to have is given on page 309). Furthermore, if one wants to study extensions of the Manakov systems, these extensions must also be Hamiltonian. This is not always the case.

Another tool at our disposal when it comes to study integrable systems is called **Painlevé analysis**. Painlevé analysis is a method that characterizes the integrability of a system through the singularities of its solutions. A singularity is a point at which a solution behaves in a irregular way. Black holes are well-known examples of singularities in the universe. In addition to Painlevé analysis, the project requires the use of the mathematical tool called **Evans function** [2]. This method was initially introduced to study the stability of soliton solutions in contexts other than optical fibers. The Evans function is a tool which can be used to indirectly find information about the eigenvalue and thus about stability (there is a great amount of literature on this subject; see [37] for a review).

The main results that my idea is based on are given in [3, 4, 14]. In [14], it is shown that crucial information concerning the Evans function (namely, values of the different derivatives) can be obtained through the computation of what is called *Melnikov integrals*. Furthermore, in [3, 4], it is shown how certain types of Melnikov integrals can be computed using Painlevé analysis. Roughly speaking I thus intend to extend these works in order to develop a method that uses Painlevé analysis to retrieve information about the Evans function. It will thus give a way to study the stability properties of the solutions of the Manakov system through singularity analysis.

It is important to mention that Painlevé analysis can be extended to non-integrable generalizations of integrable systems [3–5]. In technical terms, the expansion of the general solution of a perturbation to an integrable system about a singularity involves logarithmic terms. These expansions can thus be used to extend the results obtained on the Manakov system to extensions that take into account additional physical effects. Note that the extensions studied using Painlevé analysis do not need to be Hamiltonian.

To finish I would like to point out that I have an extensive experience in working in both fields of the study of stability properties of solutions to differential equations [6, 25–28] and the singularity analysis of the solutions [1, 7–9, 11–13, 16–21, 23, 24, 33–36]. The proposal requires someone who is proficient in both of these fields of research.

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Budget and Timetable

From July 13 through August 15, I will be working on the proposed project. Salary support of $4000 for the five-week period of July 13 to August 15 is requested.

Other support for this project

I hold an NSF grant which provides $88,444 for the period between June 30 2005 and June 30 2008 (Note that I will request a one-year no-cost extension of the grant). Although it provides a budget for travel expenses and equipment purchase related to this project, it does not provide a salary for the period during the period describe in the timetable above.

I would also like mention that, as part of of a package awarded to me when I was hired, the department of mathematics offered an amount of money equivalent to one month period of summer salary. I never used this amount until now (since my NSF grant paid my 2005, 2006, and 2007 summer salaries). Since my NSF grant is terminating this summer, I am planning to use this package as a salary on a one-month period which will not overlap with the period describe in the timetable above.