REVIEW FOR EXAM #2 MATH 323

1) Find the general solution of the equations

a) $Y^{(4)} - 7\,y''' + 17\,y'' - 17\,y' + 6y = 0$ (e^x and $x\,e^x$ are solutions).

b) $Y^{(4)} - 4\,y''' + 7\,y'' - 16\,y' + 12\,y = 0$ ($\sin(2\,x)$ is a solution).

2) Use the reduction of order to find a second solution

$$t^2\, y'' + 2\, t\, y' - 2\, y = 0, \quad y_1 = t$$

3) The differential equation

$$x^2 y'' + xy' - \left(\frac{1}{4} + x^2\right) y = 0$$

has linearly independent solutions $y_1(x) = \dfrac{e^{-x}}{\sqrt{x}}$ and $y_2(x) = \dfrac{e^{x}}{\sqrt{x}}$ for $x > 0$.

(1) Find the Wronskian of the two solutions above.

(2) Use the method of variation of parameters to find a particular solution of

$$x^2 y'' + xy' - \left(\frac{1}{4} + x^2\right) y = x^{5/2},$$

on $(0, \infty)$. (Use the formulae seen in Section 6.4 to calculate v_1' and v_2'. To calculate W_1 for example, take the Wronskian and replace the first column by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Remember that the coefficient of the highest derivative must be 1 before using the formulae.)

Hint to make sure you are on the right track: the Wronskian is $2/x$.

4) Find the values of α such that the functions $y_1 = x^2 - \alpha x + 1$, $y_2 = x^2 + x$, and $y_3 = 1$ are linearly dependent.

5) In the old dingy Ziglin Bros. Circus, one of the main attractions is Capt. Adamo, the Human Cannonball. The attraction consists in pushing Capt. Adamo out of the replica of a cannon with added smoke and sound effects. Capt. Adamo has to fly in the air for over 150 feet. The mechanism consist of a big spring pushing a board where Capt. Adamo stands. In order to achieve the desired jump, Capt. Adamo has to leave the cannon with a velocity of 100 ft per second. Over the years Capt Adamo has put on some weight and the spring is not what it used to be. The last show was a near disaster, Capt. Adamo missed the rescue net and crashed on Bobo the Clown. Fortunately, the crowd thought that it was a part of the act and the show went on.

Ziglin Bros. needs your help to recalibrate the spring. After multiple detailed analysis you conclude that the position of the board on the spring satisfies the differential equation:

$$mx'' + kx = 0$$

The initial condition is $x(0) = -L$ (the spring is compressed) and $x'(0) = 0$ (there is no initial velocity). The problem is to find the correct initial compression of the spring L.

(1) Solve the equation for the initial value.
(2) Find the time where the velocity is first maximal (the velocity is given by x').
(3) What is the maximum velocity?
(4) Express L as a function of the parameter k and m knowing that Capt. Adamo must reach a velocity of 100 ft per second.
(5) What is L in the particular case where where $k = 25$ lb/ft and Capt. Adamo weights 320 lb. ($g = 32$ lb/slug) ?

6) Consider the damped-forced oscillator

$$y'' + 2y' + 3x = \cos(\omega t).$$

(1) Find the general solution of the homogeneous problem.
(2) Find a particular solution y_p.
(3) Find the amplitude of y_p.

7) Consider the following oscillators. What is the form of the steady-state solutions? (**you do not need to find the value of the coefficients**). Recall that, in physical terms, the steady-state is the part of the solution that does not decay to zero. In mathematical term, the steady-state is the particular solution found using the undetermined coefficient method.

a) $y'' + 25\,y = \cos(3\,t)$

b) $2\,y'' + 128\,y = \sin(8\,t)$

8) Consider the mass-spring system governed by

$$y'' + b\,y' + 16\,y = 0.$$

In both cases $b = 6$ and $b = 10$, say if the system is overdamped or underdamped and write down the solution satisfying $y(0) = 1$ and $y'(0) = 0$. In the underdamped case, write the solution in the *amplitude-phase* form.

Vocabulary: the *amplitude-phase* form means the form $A\,e^{\alpha\,t}\,\sin(\beta\,t + \phi)$.