Give the order and say whether the following ode's are linear or nonlinear (L or NL).

	Equation	Order	L or NL	
Ex.	$\frac{d^2y}{dt^2} + ty\frac{dy}{dt} = 0$	2	NL	
1	$\frac{dy}{dx} = 5$	1	L	
2	$\frac{d^2y}{dx^2} + y = 6$	2	L	
3	$\frac{dy}{dx} = -y\ln y$	1	NL	
4	$\frac{d^2x}{dt^2} + .2\frac{dx}{dt} + x = 0$	2	L	
5	$\frac{d^4y}{dt^4} - ty = 0$	4	L	

Solve the initial value problem

$$\frac{dy}{dx} - 2xy = 0 \qquad y(0) = 1$$

$y(x) = e^{x^2}$

Hint. You will find $y(\sqrt{\ln 2}) = 2$

$$\frac{dy}{y} = 2x\,dx$$

$$\ln|y| = x^2 + C$$

$$y = K e^{x^2}$$

$$y(0) = 1 \rightarrow K = 1$$

Say whether the following ode's are linear or nonlinear (L or NL), exact or not (E or NE), separable or not (S or NS).

	Equation	L or NL	E or NE	S or NS
1	$t^2 \frac{dy}{dt} + ty = t$	L	NE	S
2	$3t = e^t \frac{dy}{dt} + y \ln t$	L	NE	NS
3	$y^2\, dx = -(2xy + \cos y)\, dy$	NL (*)	E	NS
4	$y\, dx = x\, dy$	L	NE	S
5	$3r = \frac{dr}{d\theta} - \theta^3$	NL	NE	NS
6	$\frac{dy}{dx} = \frac{3x^2+4x+2}{2y+1}$	N L	NE (*)	S

* Notes: 3 is nonlinear if y is the dependent variable
linear if x is the dependent variable
so, the two answers L or NL apply.

6 If written like

$$dy - \left(\frac{3x^2+4x+2}{2y+1}\right)dx = 0, \text{ it is NE}$$

If written like

$$(2y+1)dy - (3x^2+4x+2)dx = 0, \text{ it is Exact}$$

Again, here, both E and NE are correct answers.

Solve the following differential equations

$$(x+1)\,dy = (\cos x - y)\,dx$$

Exact $\Rightarrow$ $\dfrac{\partial F}{\partial x} = y - \cos(x)$ (1)

$\dfrac{\partial F}{\partial y} = x+1$ (2)

From (1) $F = xy - \sin(x) + C_1(y)$

Plug in (2) $x + C_1' = x+1 \rightarrow C_1(y) = y + C$

$\rightarrow F = xy + y - \sin(x) + C$

$$\boxed{xy + y - \sin(x) = C}$$

$$x\frac{dv}{dx} = \frac{1-4v^2}{3v}$$

Separable: $\dfrac{dx}{x} = \dfrac{3v\,dv}{1-4v^2} \longrightarrow \ln|x| + C = -\dfrac{3}{8}\ln|1-4v^2|$

$-\dfrac{8}{3}\ln|x| + C_1 = \ln|1-4v^2|$

$1-4v^2 = C_2\,x^{-8/3}$

$$\boxed{v = \frac{1}{2}\sqrt{1-C_2x^{-8/3}}}$$

$$y' = x^3(1-y),\ \ y(0) = 3$$

separable $\dfrac{dy}{1-y} = x^3\,dx$

$-\ln|y| = \dfrac{x^4}{4} + C$

$y = Ke^{-x^4/4}$

$y(0) = 3 \rightarrow K = 3 \longrightarrow$ $\boxed{y = 3e^{-\frac{x^4}{4}}}$

$$\frac{dy}{dx} + 2xy = x$$

Integrating factor: $e^{x^2} \rightarrow \frac{d}{dx}\left(e^{x^2} y\right) = x e^{x^2}$

$$e^{x^2} y = \frac{1}{2} e^{x^2} + C$$

$$\boxed{y = \frac{1}{2} + \frac{C}{e^{x^2}}}$$

$$\cos x \, \frac{dy}{dx} + y \, \sin x = 2 \, x \, \cos^2 x$$

Integrating factor

$$\frac{dy}{dx} + \frac{\sin x)}{\cos(x)} y = 2x \cos(x)$$

Integrating factor: $e^{\int \frac{\sin x}{\cos x} dx} = e^{-\ln|\cos x|} = \frac{1}{|\cos x|}$

$$\rightarrow \frac{d}{dx}\left(\frac{y}{\cos x}\right) = 2x \rightarrow \boxed{y = x^2 \cos(x) + C}$$

$$(1/y)\, dx + (3y - x/y^2)\, dy = 0$$

Exact

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= \frac{1}{y} \\ \frac{\partial F}{\partial y} &= 3y - \frac{x}{y^2} \end{aligned}\right\} \quad F = \frac{3y^2}{2} + \frac{x}{y} + C$$

$$\rightarrow \boxed{\frac{3y^2}{2} + \frac{x}{y} = C}$$

Note: forget about

$$(ye^{xy} - 1/y)dx + (2yx^2 - cosy)\, dy = 0, \quad y(1) = \pi$$

The population $P(t)$, measured in units of 1000, of a species obeys the ode

$$\frac{dP}{dt} = 2P(1-P)$$

where time t is measured in years.

If the starting population was 2000, i.e. $P(0) = 2$, solve for

$$P(t) = \frac{-2e^{2t}}{1-2e^{2t}} = \frac{-2}{e^{-2t}-2} = \frac{2}{2-e^{-2t}}$$

[Check your answer: *Hint. You will find* $P(\ln 2) = \frac{8}{7}$]

separable:

$$\frac{dP}{P(1-P)} = 2dt$$

Partial fractions: $\frac{1}{P(1-P)} = \frac{1}{P} + \frac{1}{1-P}$

$$\rightarrow \int \frac{dP}{P(1-P)} = \ln|P| - \ln|1-P| = \ln\left|\frac{P}{1-P}\right|$$

$$\rightarrow \ln\left|\frac{P}{1-P}\right| = 2t + C \rightarrow \frac{P}{1-P} = Ke^{2t}$$

$$\rightarrow P = (1-P)Ke^{2t}$$

$$\rightarrow P = \frac{Ke^{2t}}{1+Ke^{2t}} \qquad P(0) = 2 \rightarrow \frac{K}{1+K} = 2 \rightarrow K = -2$$

To which value does $P(t)$ tend as $t \to \infty$?

$$\lim_{t\to\infty} P(t) = 1$$

No need to solve, just look at the slope field to find $\lim_{t\to\infty} P(t)$

The amount of radioactive isotope $Q(t)$ (measured in kgs. or as a percentage of its original value) obeys the ode

$$\frac{dQ(t)}{dt} = -kQ(t)$$

where t, time is measured in years. Given $Q(0) = Q_0$, find $Q(t)$ in terms of Q_0, k and t.

$$\boxed{Q(t) = Q_0 e^{-kt}}$$

The half-life τ is the time it takes the amount $Q(t)$ to decay to $\frac{1}{2}Q_0$; i.e. $Q(\tau) = \frac{1}{2}Q_0$. Find an expression for τ in terms of k, the decay rate

$$\boxed{\tau = -\frac{1}{K}\ln\left(\frac{1}{2}\right)}$$

$$Q_0 e^{-K\tau} = \frac{1}{2}Q_0 \rightarrow e^{-K\tau} = \frac{1}{2}$$

$$\rightarrow \tau = -\frac{1}{K}\ln\left(\frac{1}{2}\right)$$

A bone is found on an archeological dig. When found, the amount of radioactive isotope C^{14} (carbon 14) was 25% of what it was when the bone was part of a living organism. Given the half life of C^{14} is 5600 years, how long has the bone been dead (age)?

$$\boxed{Age = 11200}$$

First find K: From above $\tau = -\frac{1}{K}\ln\left(\frac{1}{2}\right)$ $\tau = 5600$ yrs

$$\rightarrow K = -\frac{1}{5600}\ln\left(\frac{1}{2}\right)$$

Let t be the time when only 25% remains

$$\rightarrow Q_0 e^{-Kt} = 0.25\, Q_0$$

$$\rightarrow t = -\frac{1}{K}\ln(0.25) = \frac{1}{\frac{1}{5600}\ln\left(\frac{1}{2}\right)}\ln(0.25) = 11200$$

Consider the ode

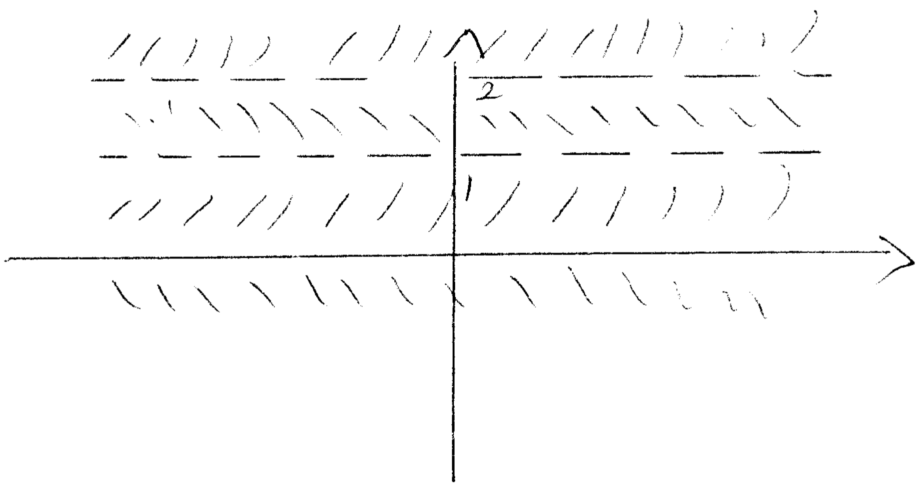
$$\frac{dy}{dt} = y(1-y)(2-y)$$

Given (a) $0 < y(0) < 1$, $y(t) \to$ $\boxed{1}$ as $t \to \infty$.

Given (b) $1 < y(0) < 2$, $y(t) \to$ $\boxed{1}$ as $t \to \infty$.

Given (c) $y(0) > 2$, $y(t) \to$ $\boxed{\infty}$ as $t \to \infty$.

Obtained by studying the slope field below



Note: this is a very approximati version of the slope field!

Model for an object falling toward earth. Assuming that only air and gravity are acting on the object, the velocity v is given by

$$m\frac{dv}{dt} = m\,g - bv$$

where m is the mass of the object, g is the acceleration due to gravity and $b > 0$ is constant. If $m = 100kg$, $g = -9.8m/sec^2$, $b = 5kg/sec$, $v(0) = 10m/sec$, solve for $v(t)$. What is the terminal velocity?

$$100\,\frac{dV}{dt} = -100\cdot 9.8 - 5V$$

$$\frac{dV}{dt} = -\left(\frac{5V}{100} + 9.8\right)$$

$$= -\frac{5}{100}\left(V + \frac{980}{5}\right) = -0.05\left(V + 196\right)$$

$$\frac{dV}{V+196} = -\frac{5}{100}\,dt = -0.05\,dt$$

$$\rightarrow V = Ke^{-0.05t} + -196$$

$$V(0) = 10 \Rightarrow K = 206$$

$$\Rightarrow \boxed{V = 206e^{-0.05t} - 196}$$

Terminal velocity corresponds to the limit $t \to \infty$

One can get it by finding the stationary solutions:

$$mg - bV = 0$$

$$-980 - 5V = 0$$

$$V = -\frac{980}{5} = \boxed{-196}$$

Newton's law of cooling:

$$\frac{dH}{dt} = k\,(T - H)$$

where H is the temperature of the object, $k > 0$ is constant, and T is the temperature of the surrounding.

Solve for H.

A thermometer reading 100°F is placed in a medium having a constant temperature of 70°F. After 6 minutes, the thermometer reads 80°F. What is the reading after 20 minutes?

We all know how to solve this $\boxed{H = Ce^{-kt} + T}$

$T = 70, \quad H(0) = 100$

$\Rightarrow \quad H = 30e^{-kt} + 70$

$H(6) = 80 = 30e^{-k6} + 70$

$\rightarrow \quad k = -\frac{1}{6}\ln\left(\frac{1}{3}\right)$

$\rightarrow \quad H(20) = 30e^{\frac{20}{6}\ln\left(\frac{1}{3}\right)} + 70$

$= 30\left(\frac{1}{3}\right)^{\frac{10}{3}} + 70 \approx 70.77$